Phys 371
Fall 2019
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Lecture 11, Compton Scattering

Friday 20 September, 2019

Goal: To construct a consistent system of Relativistic Dynamics

Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value *c* in every direction in all inertial reference frames.

The first postulate implies that the laws of physics are the same for all inertial observers.

Therefore we need to formulate the laws of a physics in a way that transform properly under the Lorentz Transformation (LT)

The 4-vector description of an event transforms under an LT as:

$$\boldsymbol{x}'^{(4)} = \bar{\bar{\boldsymbol{\Lambda}}}\,\boldsymbol{x}^{(4)} \quad \textbf{where} \quad \boldsymbol{x}^{(4)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \textbf{and} \quad \bar{\bar{\boldsymbol{\Lambda}}} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Now we want to formulate dynamical variables that transform the same way – as 4-vectors

Start with the law of conservation of momentum

Last week we showed that $\vec{p} = m\vec{u}$ is not conserved in relativistic collisions
Go back to the drawing board!

World Line for a particle

Think of $x^{(4)} = (\vec{x}(t), ct)$ as the trajectory of a point particle in 4-dimensional space-time (i.e. a "world line"). It represents a series of "events" corresponding to the instantaneous location of the particle.

A 4-vector velocity can be formulated as: $u^{(4)} = \frac{dx^{(4)}}{dt_0} = \gamma(u)(\vec{u}, c)$.

Where $dt_0 = dt/\gamma(u)$ is the proper time differential as measured in the rest frame of the particle, and $\gamma(u) = 1/\sqrt{1-(u/c)^2}$ is the γ-factor associated with particle's velocity $\vec{u}$ as measured in a given reference frame.

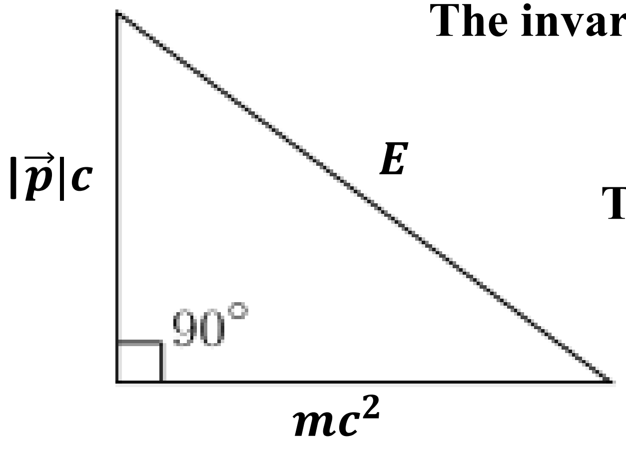
The 4-vector momentum is defined as $\boldsymbol{p^{(4)} = m\gamma(u)(\vec{u}, c)}$,

The fourth component is defined as the relativistic energy $\boldsymbol{E/c}$: $\boldsymbol{p^{(4)} = (m\gamma(u)\vec{u}, E/c)}$

$\boldsymbol{E = \gamma(u)mc^2}$ for a free particle (i.e. a particle subject to zero net external force).

Comparing two forms of the four-momentum, namely $\boldsymbol{p^{(4)} = m\gamma(u)(\vec{u}, c)}$ and $\boldsymbol{p^{(4)} = (\vec{p}, E/c)}$

One finds: $\boldsymbol{\vec{u} = \frac{\vec{p}c^2}{E}}$ which can also write $\boldsymbol{\vec{u} = \vec{p}/(\gamma(u)m)}$



The invariant length of $\boldsymbol{p^{(4)}}$ gives: $\boldsymbol{\vec{p}\cdot\vec{p} - (E/c)^2 = -(mc)^2}$

The useful relation: $\boldsymbol{E^2 = (\vec{p}c)^2 + (mc^2)^2}$

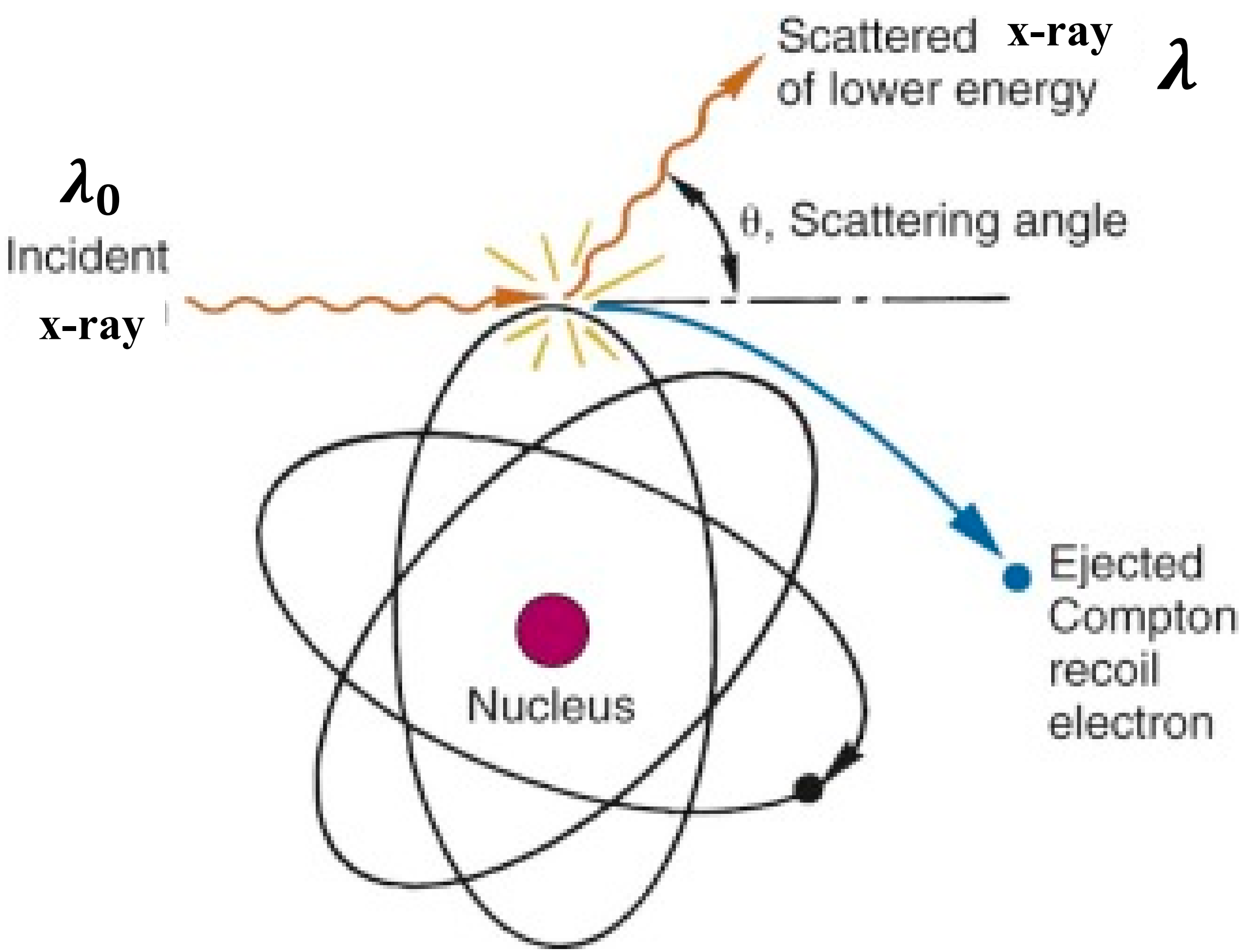
By design, the four-momentum transforms according to the Lorentz transformation. In other words, if a particle is observed to have four momentum $\boldsymbol{p}^{(4)} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ E/c \end{pmatrix}$ in reference frame S, it will have four-momentum $\boldsymbol{p}'^{(4)} = \begin{pmatrix} p_1{}' \\ p_2{}' \\ p_3{}' \\ E'/c \end{pmatrix}$ as witnessed in reference frame S' which are related through a Lorentz transformation corresponding to a boost by speed V along the common $x - x'$ axis: $\boldsymbol{p}'^{(4)} = \overline{\overline{\Lambda}}\,\boldsymbol{p}^{(4)}$ with $\overline{\overline{\Lambda}} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$ and $\beta = V/c$, and $\gamma = \frac{1}{\sqrt{1-(V/c)^2}}$.

This transformation is a rotation in momentum-energy 4-space and mixes momentum and energy (divided by c) together.

Compton Scattering

An example of the use of relativistic 4-momentum

Scattering of light (x-ray) by a (nominally) stationary electron



What is measured:

1) **Wavelength of incident light (λ_0)**
2) **Scattering angle of light (θ)**
3) **Wavelength of the scattered light (λ)**

Objective:

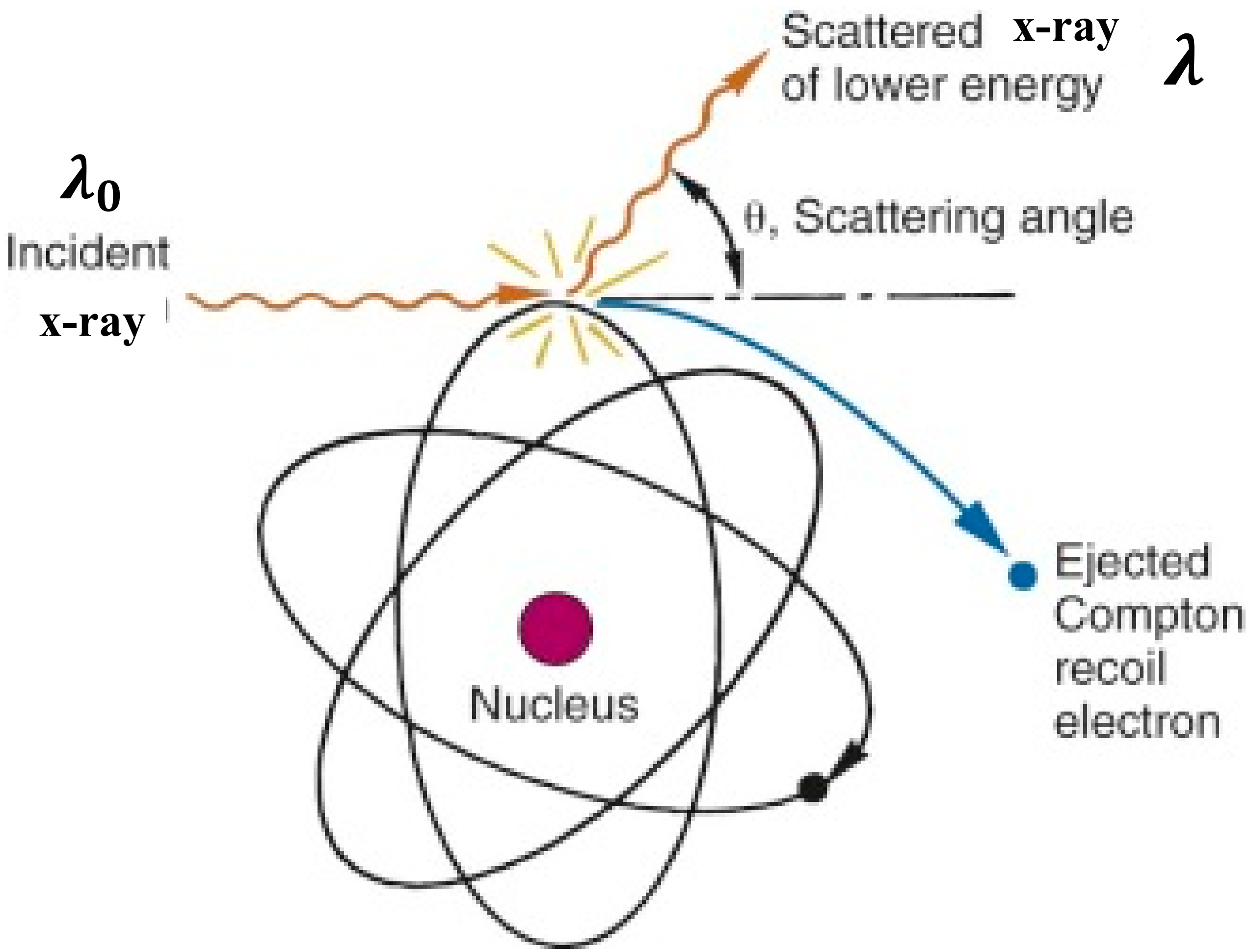
Figure out the relationship between λ, λ_0, and θ.

Approach:

Conservation of 4-momentum

Compton Scattering

Scattering of light (x-ray) by a (nominally) stationary electron



Compton observed $\lambda > \lambda_0$ whenever $\theta \neq 0$

and

a classical treatment of this process was not successful!

How do we write the 4-momentum of the x-ray?
Treat the "x-ray particle" as a mass-less particle.

The useful relation $E^2 = (\vec{p}c)^2 + (mc^2)^2$ now becomes $E = pc$, where p is the magnitude of the 3-momentum of the photon

We know that $\vec{u} = \vec{p}c^2/E$, hence $u = c$, thus light always travels at the speed of light!

So if the "x-ray particle" is massless, what is it's momentum, and energy?

Now for something completely different!

Einstein made this proposal (in a different context!):
$E = \hbar\omega$, where h is Planck's constant ($\hbar \equiv h/2\pi$)

He proposed that the x-ray interacts with the electron as if it were a particle!
Treat the scattering process as a particle-particle interaction

An electromagnetic wave satisfies: $f\lambda = c$

Defining $k = 2\pi/\lambda$ and $\omega = 2\pi f$, we can also write this as $\omega = kc$
(the dispersion relation for light)

For a mass-less particle, the useful relation $E^2 = (\vec{p}c)^2 + (mc^2)^2$ reduces to:

$$E = pc$$

But Einstein also said that $E = \hbar\omega$. Equating these two expressions we can solve for the momentum of the x-ray: $p = \hbar\omega/c$.

Now use the fact that $\omega = kc$ to re-write the momentum as $p = \hbar k$, or in full vector form as $\vec{p} = \hbar\vec{k}$

Thus light (like an x-ray) carries both momentum and energy
To commemorate this remarkable result, the *particle-like* property of light is called a "Photon"

We can finally write down-down the 4-vector for a photon

$$p_\gamma^{(4)} = \hbar\left(\vec{k}, \frac{\omega}{c}\right) = \frac{\hbar\omega}{c}(\hat{k}, 1)$$

Momentum direction unit vector

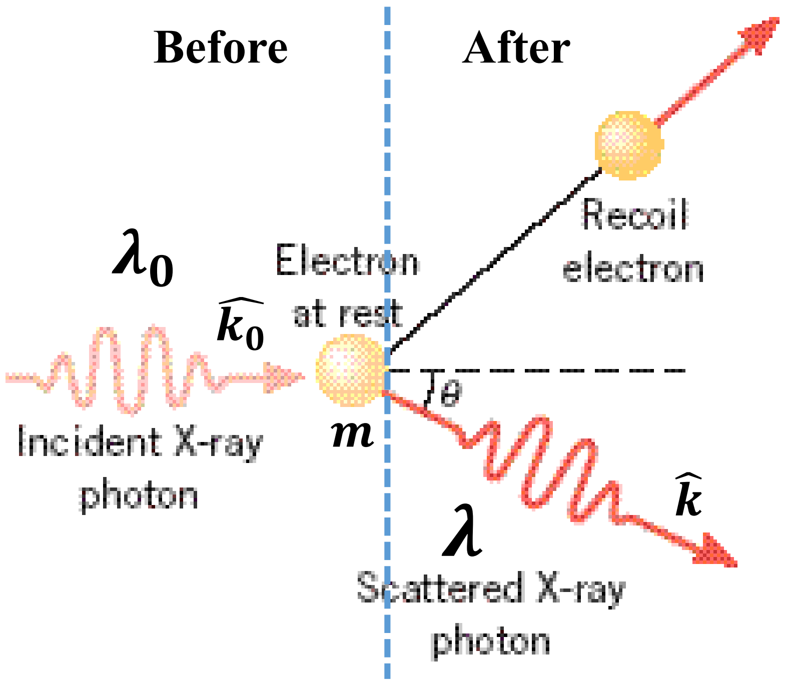
4-vector for a photon $\boldsymbol{p_\gamma^{(4)}} = \hbar\left(\vec{\boldsymbol{k}}, \frac{\boldsymbol{\omega}}{\boldsymbol{c}}\right) = \frac{\hbar\boldsymbol{\omega}}{\boldsymbol{c}}(\widehat{\boldsymbol{k}}, \mathbf{1})$

Look at the invariant length of this 4-vector: $\boldsymbol{p_\gamma^{(4)}} \cdot \boldsymbol{p_\gamma^{(4)}}$

$$\boldsymbol{p_\gamma^{(4)}} \cdot \boldsymbol{p_\gamma^{(4)}} = \left(\frac{\hbar\boldsymbol{\omega}}{\boldsymbol{c}}\right)^{\mathbf{2}} \left[\widehat{\boldsymbol{k}} \cdot \widehat{\boldsymbol{k}} - \mathbf{1}^{\mathbf{2}}\right] = \mathbf{0}$$

This handy property suggests that "squaring" the momentum conservation equation may be a fruitful way to solve the equations

Compton Scattering



Write down the total 2-particle 4-momentum Before and After the collision
Equate them and solve for λ in terms of λ_0 and θ

$$p^{(4)}_{Before} = p^{(4)}_{\gamma_0} + p^{(4)}_{0}$$

$$p^{(4)}_{Before} = \frac{\hbar\omega_0}{c}(\widehat{k_0}, 1) + (0, 0, 0, mc)$$

$$p^{(4)}_{After} = p^{(4)}_{\gamma} + p^{(4)}_{electron}$$

$$p^{(4)}_{After} = \frac{\hbar\omega}{c}(\hat{k}, 1) + p^{(4)}$$

Conservation of 4-Momentum in Compton Scattering

$$p^{(4)}_{Before} = p^{(4)}_{After}$$

$$p^{(4)}_{\gamma_0} + p^{(4)}_0 = p^{(4)}_\gamma + p^{(4)}$$

Manipulate it to isolate $p^{(4)}$ on the RHS,

$$p^{(4)}_0 + \left(p^{(4)}_{\gamma_0} - p^{(4)}_\gamma\right) = p^{(4)}$$

then "square" the equation

$$p^{(4)2} \equiv p^{(4)} \cdot p^{(4)}$$

$$\cancel{p^{(4)2}_0} + 2p^{(4)}_0 \cdot \left(p^{(4)}_{\gamma_0} - p^{(4)}_\gamma\right) + \left(p^{(4)}_{\gamma_0} - p^{(4)}_\gamma\right)^2 = \cancel{p^{(4)2}}$$

Note that for the electron: $p^{(4)2}_0 = p^{(4)2} = -(mc)^2$ so that those terms cancel!

$$2p_0^{(4)}\cdot\left(p_{\gamma_0}^{(4)}-p_\gamma^{(4)}\right)+\underbrace{{p_{\gamma_0}^{(4)}}^2}_{0}-2p_{\gamma_0}^{(4)}\cdot p_\gamma^{(4)}+\underbrace{{p_\gamma^{(4)}}^2}_{0}=0$$

Note that for the photon: ${p_\gamma^{(4)}}^2 = 0$, so those terms go away!

So now we have $$p_0^{(4)}\cdot\left(p_{\gamma_0}^{(4)}-p_\gamma^{(4)}\right)=p_{\gamma_0}^{(4)}\cdot p_\gamma^{(4)}$$

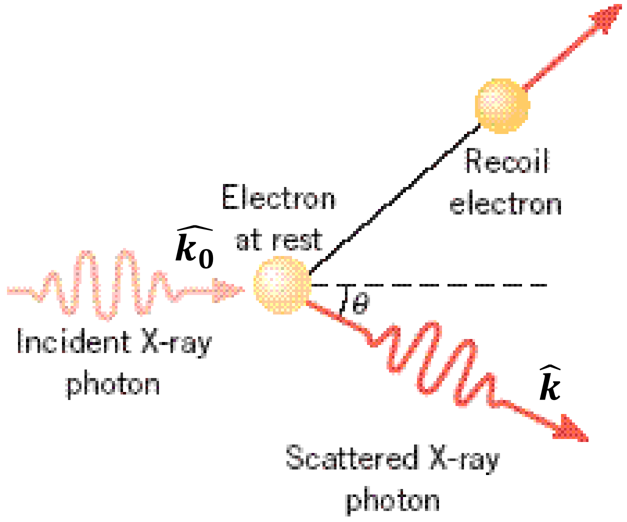
Plug in the values of the 4-momenta:

$$(0,0,0,mc)\cdot\left(\frac{\hbar\omega_0}{c}\widehat{k_0}-\frac{\hbar\omega}{c}\widehat{k},\frac{\hbar}{c}(\omega_0-\omega)\right)=\frac{\hbar\omega_0}{c}\left(\widehat{k_0},1\right)\cdot\frac{\hbar\omega}{c}\left(\widehat{k},1\right)$$

Simplify these scalar products to an algebraic equation:

$$-mc\frac{\hbar}{c}(\omega_0-\omega)=\frac{\hbar\omega_0}{c}\frac{\hbar\omega}{c}\left(\widehat{k_0}\cdot\widehat{k}-1^2\right)$$

Note that $\widehat{k_0}\cdot\widehat{k}=\cos\theta$



Simplify to $$\omega_0 - \omega = \frac{\hbar}{mc^2}\omega_0\omega(1-\cos\theta)$$

Divide through by $\omega_0\omega$ to find

$$\frac{1}{\omega} - \frac{1}{\omega_0} = \frac{\hbar}{mc^2}(1-\cos\theta)$$

Use the fact that $\omega = kc = 2\pi c/\lambda$ to find

$$\lambda - \lambda_0 = \frac{h}{mc}(1-\cos\theta)$$

The Compton wavelength $\lambda_C = \frac{h}{mc}$ sets the scale for the effect.

Multiplying top and bottom by a factor of c one has $\lambda_C = \frac{hc}{mc^2} = \frac{1239.8\ eV{-}nm}{0.511x10^6\ eV} = 2.43\ pm$, which is a γ-ray wavelength. One needs a good x-ray spectrometer to see the shift in wavelengths.

Thanks for Listening!!!